Constructing Operator Valued Probability Measures in Phase Space

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Abstract

Probability measures (quasi probability mass), given in the form of integrals of Wigner function over areas of the underlying phase space, give rise to operator valued probability measures (OVM). General construction methods of OVMs, are investigated in terms of geometric positive trace increasing maps (PTI), for general 1D domains, as well as 2D shapes e.g. circles, disks. Spectral properties of OVMs and operational implementations of their constructing PITs are discussed.

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An OVM is a map $K : \mathcal{F} \rightarrow \mathcal{L}(H)$, from a $\sigma-$algebra $\mathcal{F}$ of subsets of an nonempty set Ω, to the bounded operators $\mathcal{L}(H)$ on a Hilbert space H, such that for $\Psi \in H$, and $X \in \mathcal{F}$, the function $\mu_\Psi(X) \equiv \langle\Psi|K(X)\Psi\rangle$, is a normalized (i.e. $\mu_\Psi(\Omega) = 1$), generalized (i.e. negative valued) measure (i.e. $\sigma-$additive set function). If $F(X) \geq 0$, i.e. $\mu_\Psi(X) > 0$, $\forall X \in \mathcal{F}$ we have a positive OVM, further if $K(X)^\dagger = K(X)$, and $K(X)^2 = K(X)$, we have a projective OVM, namely an observable (see e.g. [1]). For the case of Wigner function[2][3] $W_{|\psi>}(\alpha) = \langle\Psi|D(\alpha)\Pi D(\alpha)^\dagger|\Psi\rangle$, we have $\Omega \equiv C$, $H \equiv$span$(|n\rangle, n = 0, 1, ...)$ the Fock space, with N the number operator, $\Pi = e^{i\pi N}$ the parity operator, and $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{(iqP - ipQ)}$ the displacement operator. Integrals of Wigner function[4][5] $\int_X W_{|\psi>}(\alpha)d^2\alpha = Tr(|\Psi\rangle\langle\Psi|K(X))$, physically define the *quasi probability mass* over X, in terms of OVM $K(X) \equiv \int_X D(\alpha)\Pi D(\alpha)^\dagger d^2\alpha$.

Proposition 1 *Let X_q a region in q - axis determined by the characteristic function (cfun), $\chi(q), q \in R$, with Fourier transform (FT) $\widetilde{\chi}(p)$, then the associated OVM is $K_q = \widetilde{\chi}(P)\Pi$. The solution of eigen problem $K_q|\Psi_\pm\rangle = \lambda_\pm|\Psi_\pm\rangle$, determines the eigenvalues $\lambda_\pm(p) = \pm\widetilde{\chi}(p)$, and the eigenvectors $|\Psi_\pm\rangle = |p\rangle \pm |-$*

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$p\rangle$. *If further* $\chi(q)$ *is* $L-$*periodic with Fourier series* $\chi(q) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(\frac{m\pi q}{L}) + b_m \sin(\frac{m\pi q}{L})$, *the OVM becomes*

$$K_q = \Pi \left[\frac{\pi a_0}{2} |p=0\rangle\langle p=0| + \pi \sum_{m=1}^{\infty} r_m \left(e^{i\phi_m} |p = \frac{m\pi}{L}\rangle\langle p = \frac{m\pi}{L}| \right.\right.$$
$$\left.\left. + e^{-i\phi_m} |p = -\frac{m\pi}{L}\rangle\langle p = -\frac{m\pi}{L}| \right) \right], \quad (1)$$

where $r_m \equiv |a_m + ib_m|$ *,and* $\phi_m \equiv \arg(a_m + ib_m)$.

Proof: The *region operator* (convention: alternative name for OVM related to quasi-probability functions, see below), is constructed by smearing along $q-$axis the point at the origin as follows

$$K_q = \int_{X_q} e^{i\frac{qP}{2}} \Pi e^{-i\frac{qP}{2}} dq = \int_R \chi(q) e^{iqP} dq \Pi = \int_R \widetilde{\chi}(p) |p\rangle\langle -p| dp \equiv \widetilde{\chi}(P)\Pi, \quad (2)$$

where $\widetilde{f}(p) = \int_R f(q) e^{iqp} dq$, is the FT. Straightforward evaluation of FT of the cfun of eq. (3), below leads to the result. The eigen problem is also easily solved utilizing the action of parity operator on momentum state.■

Remarks: 1) From FT of a $L-$periodic cfun of some region we obtain the eigenvalues of its associateed OVM $\lambda_{\pm}(p) = \pm\widetilde{\chi}(p)$, where

$$\widetilde{\chi}(w) = a_0 \pi \delta(w)$$
$$+ \pi \sum_{m=1}^{\infty} \left[a_m \left(\delta\left(w - \frac{m\pi}{L}\right) + \delta\left(w + \frac{m\pi}{L}\right) \right) - ib_m \left(\delta\left(w - \frac{m\pi}{L}\right) - \delta\left(w + \frac{m\pi}{L}\right) \right) \right]. \quad (3)$$

If cfun is asymmetric then its FT is complex valued, this implies the region operator would not be a real one.

2) Let e.g. the pure density matrix $\rho = |\Psi\rangle\langle\Psi|$, determined by an even parity state vector i.e. $P|\Psi\rangle = |\Psi\rangle$, then the occupation probability of the region X_q is

$$p_\rho(X_q) = Tr(K_q \rho) = \int_R \widetilde{\chi}(p) |\Psi(p)|^2 dp. \quad (4)$$

If further the region X_q is symmetric with respect to the origin, i.e. its cfun is an even function (i.e. $b_m = 0$), then we obtain the probability

$$p_\rho(X_q) = a_0 \pi |\Psi(0)|^2 + 2\pi \sum_{m=1}^{\infty} a_m |\Psi(\frac{m\pi}{L})|^2. \quad (5)$$

3) Let us choose the special region for which $a_m \geq 0$, $b_m = 0$, then the associated region operator becomes

$$K_q = a_0 \pi |p=0\rangle\langle p=0| + \pi \sum_{m=1}^{\infty} a_m \left(|p = -\frac{m\pi}{L}\rangle\langle p = \frac{m\pi}{L}| \right.$$
$$\left. + |p = \frac{m\pi}{L}\rangle\langle p = -\frac{m\pi}{L}| \right). \quad (6)$$

This is also expressed as $K_q = \varepsilon_q(|p=0\rangle\langle p=0|)$, with ε_q been a PTI map is generated by the set of Kraus generators[6] $\varepsilon_q \equiv \left\{\sqrt{a_0\pi}\mathbf{1}, \sqrt{a_m}e^{-i\frac{m\pi}{L}Q}, \sqrt{a_m}e^{i\frac{m\pi}{L}Q}\right\}_{m=1}^{\infty}$. The latter expression of K_q, if utilized together with state-observable duality (c.f. [7]), would provide, by means of a unitarization of ε_q map, means for some physical implementation of the general region operator K_q (see similar constructions of other region operators in [8], and below).

4) The operator K_q defined on a region of $q-$axis with cfun $\chi(q)$, can be rotated by $\frac{\pi}{2}$ radians to become K_p, i.e. a similar operator on p-axis which reads

$$K_p \equiv e^{i\frac{\pi}{2}N}K_q e^{-i\frac{\pi}{2}N} = e^{i\frac{\pi}{2}N}\widetilde{\chi}(P)\Pi e^{-i\frac{\pi}{2}N} = \widetilde{\chi}(Q)\Pi. \tag{7}$$

Having available operators K_q, K_p, been defined along orthogonal axes, we further rotate them clockwise by an angle θ, to obtain the respective operators along new rotated axes as follows,

$$K_q^{\theta} = e^{i\theta N}K_q e^{-i\theta N} = e^{i\theta N}\widetilde{\chi}(P)\Pi e^{-i\theta N} = \widetilde{\chi}(\cos\theta P - \sin\theta Q)\Pi, \tag{8}$$

and

$$K_p^{\theta} = e^{i\theta N}K_p e^{-i\theta N} = e^{i\theta N}\widetilde{\chi}(Q)\Pi e^{-i\theta N} = \widetilde{\chi}(\cos\theta Q + \sin\theta P)\Pi. \tag{9}$$

These two operators are related by a $\frac{\pi}{2}$ radians rotation i.e. $K_p^{\theta} = e^{i\frac{\pi}{2}N}K_q^{\theta}e^{-i\frac{\pi}{2}N}$.

5) Using properties of Fourier transform we obtain shift transforms of a region operator e.g. $K_q e^{icP} = \widetilde{\chi}(P)\Pi$ as follows,

$$e^{icP}K_q = \widetilde{\chi}(P + c\mathbf{1})\Pi, \quad K_q e^{icP} = \widetilde{\chi}(P - c\mathbf{1})\Pi, \; e^{i\frac{c}{2}P}K_q e^{-i\frac{c}{2}P} = \widetilde{\chi}(P + c\mathbf{1})\Pi. \tag{10}$$

6) Using the squeezing operator $S(\zeta) = e^{\zeta a^{\dagger 2} - \zeta^* a^2} = e^{\frac{1}{2}r(Q^2 - P^2)}$,with parameter $\zeta = \frac{1}{2}re^{-i2\phi}$, we obtain by similarity transformation multiplicative actions on position and momentum operators

$$S(\zeta)^{\dagger}QS(\zeta) = Qe^{r}, \; S(\zeta)^{\dagger}PS(\zeta) = Pe^{-r}. \tag{11}$$

This gives rise to a region operator with (squeezed) scaled support on q-axis i.e.

$$S(\zeta)^{\dagger}K_q S(\zeta) = S(\zeta)^{\dagger}\widetilde{\chi}(P)S(\zeta)\Pi = \widetilde{\chi}(Pe^{-r})\Pi. \tag{12}$$

7) OVM from s-Parametrized Quasiprobability Functions: Working out the power series expression of $s-$parametrized quasi probabilities phase space functions $F(\alpha; s)$, of [9, 3]

$$F(\alpha; s) = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(1+s)^k}{(1-s)^{k+1}}(-1)^k\langle k|D(\alpha)^{\dagger}\rho D(\alpha)|k\rangle, \tag{13}$$

we obtain that

$$F(\alpha; s) = \frac{2}{\pi}Tr(\rho\widehat{F}(\alpha; s)) \equiv \frac{2}{\pi}Tr(\rho D(\alpha)\Pi(s)D(\alpha)^{\dagger}), \tag{14}$$

where the $s-$parametrized parity operator $\Pi(s)=\frac{(1+s)^N}{(1-s)^{N+1}}\Pi$, is now displaced by the coherent state generating operators. Special cases are the Glauber-Sudarshan P function $F(\alpha;s=1)=P(\alpha)$, the Wigner function $F(\alpha;s=0)=W(\alpha)$, and the Q positive function $F(\alpha;s=-1)=Q(\alpha)$. The "point operator" at the origin of phase plane is now the s-parametrized parity operator $\Pi(s)$. Applying appropriate geometric PTI maps $\Pi(s)$ as in the present case of Wigner function, i.e. $s=0$, we can derive s-parametrized OVMs.

Examples: We proceed with examples that elucidate the general construction: 1) Let us choose $X_q=\{1,2,...,n\}$, namely the case where the region consists of the first n positive integers. This set with cfun $\chi(x)=1$ for $x=1,2,...,n$, and 0 otherwise, leads to region operator

$$K_q=\int_R \chi(x)e^{i2xP}dx\Pi=\sum_{m=1}^{n}e^{i2mP}\Pi=\frac{e^{i(2n+1)P}-e^{iP}}{2i\sin P}\Pi. \tag{15}$$

As the set is not symmetric with respect to the origin of axes, the resulting region operator is a complex one in this case.

2) Circle OVM $K_C(a)$; construction steps: displace by a along q-axis the 0 point OVM i.e. $\Pi\to e^{-iaP}\Pi e^{iaP}$, and rotate it around by means of the continuous PTI map $\varepsilon_{2\pi}$, with Kraus generators $\{e^{-i\phi N};0\leq\phi<2\pi\}$, i.e. $e^{-iaP}\Pi e^{iaP}\to\varepsilon_{2\pi}(e^{-iaP}\Pi e^{iaP})$. Explicitly,

$$\begin{aligned}\Pi\to e^{-iaP}\Pi e^{iaP}\to\varepsilon_{2\pi}(e^{-iaP}\Pi e^{iaP})&\equiv\int_0^{2\pi}d\phi e^{-i\phi N}e^{-iaP}\Pi e^{iaP}e^{i\phi N}\\ =\Pi\int_0^{2\pi}d\phi e^{-i\phi N}e^{2iaP}e^{i\phi N}&=\Pi\sum_{mn=0}^{\infty}\int_0^{2\pi}d\phi e^{-i\phi N}|m\rangle\langle m|e^{2iaP}|n\rangle\langle n|e^{i\phi N}\end{aligned} \tag{16}$$

$$=2\pi\Pi\sum_{n=0}^{\infty}e^{-a^2/2}L_n(a^2)|n\rangle\langle n|\equiv K_C(a), \tag{17}$$

where $\langle n|e^{2iaP}|n\rangle=e^{-a^2/2}L_n(a^2)$, L_n been the Laguerre polynomials (result obtained originally in [4]).

3) Disc OVM $K_D(a)$; construction steps: start with OVM $K_L(a)$ of line segment extended along $[-L,L]$, been constructed by means of PTI map in [8], now rotate it around by means of the continuous PTI map $\varepsilon_{2\pi}$, with Kraus

generators $\{e^{-i\phi N}; 0 \leq \phi < 2\pi\}$. Explicitly,

$$K_L(a) = \frac{\sin(Pa)}{P}\Pi \to \varepsilon_{2\pi}(K_L(a)) \equiv \int_0^{2\pi} d\phi e^{-i\phi N} K_L(a) e^{i\phi N}$$
$$= \Pi \int_0^{2\pi} d\phi e^{-i\phi N} \frac{\sin(Pa)}{P} e^{i\phi N} = \int_{-a/2}^{a/2} dx \left(\Pi \int_0^{2\pi} d\phi e^{-i\phi N} e^{2ixP} e^{i\phi N} \right) \tag{18}$$
$$= 2\pi\Pi \sum_{n=0}^{\infty} \left(\int_{-a/2}^{a/2} dx e^{-x^2/2} L_n(x^2) \right) |n\rangle\langle n| = \int_{-a/2}^{a/2} dx K_C(x) \equiv K_D(a). \tag{19}$$

Operational construction of OVMs: Let $\Pi^1_\alpha = D_\alpha \Pi D^\dagger_\alpha \otimes \mathbf{1}$,$\Pi^2_\beta = \mathbf{1} \otimes D_\beta \Pi D^\dagger_\beta$, be point operators with support on points $\alpha, \beta \in C$, belonging to two respective phase planes. Their sum acting in $H \otimes H$, is $\Pi^1_\alpha + \Pi^2_\beta = D_\alpha \otimes D_\beta (\Pi^1 + \Pi^2) D^\dagger_\alpha \otimes D^\dagger_\beta$, i.e. is the displaced sum of two operators with support on the zero point of the respective phase planes. To construct the latter we resort to the property: the unitary operator $V = \exp(-i\frac{\pi}{2}J)$, where $J = \frac{1}{2i}(a^\dagger_1 a_2 - a^\dagger_2 a_1)$, serves as the permutation operators between parity operators of two different Hilbert spaces i.e. $\Pi^2 = V^2 \Pi^1 V^{\dagger 2}$. Then $\Pi^1 + \Pi^2 = \Pi^1 + V^2 \Pi^1 V^{\dagger 2} \equiv \varepsilon(\Pi^1)$, where we have introduced the positive trace increasing map ε, with two Kraus operators $(\mathbf{1}, V^2)$. Let the density matrix ρ acting in $H \otimes H$, and consider the expectation value $\langle \Pi^1 + \Pi^2 \rangle \equiv Tr(\rho(\Pi^1 + \Pi^2)) = Tr(\rho\varepsilon(\Pi^1)) = Tr(\varepsilon^*(\rho)\Pi^1)$, where the dual PTI map ε^* has been introduced (see [7]), as $\varepsilon^*(\rho) = \rho + V^{\dagger 2}\rho V^2$. Obtaining the value $\langle \Pi^1 + \Pi^2 \rangle$, requires the operational construction of density matrix $\varepsilon^*(\rho)$; which amounts to a unitary dilation of ε^*. To accomplish this we introduce the auxiliary space H_A =span$\{|0\rangle, |1\rangle\}$, and the unitary operator

$$W = \begin{pmatrix} \mathbf{1} & -V^2 \\ V^{\dagger 2} & \mathbf{1} \end{pmatrix}, \tag{20}$$

acting on $H_A \otimes H \otimes H$. Then we obtain $\varepsilon^*(\rho) = Tr_A W^\dagger (|0\rangle\langle 0| \otimes \rho) W$; the partial tracing map Tr_A signifies operationally an unconditional measurment in H_A[10].

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References

[1] P. Bush, M. Grabowski and P. Lahti, *Operational Quantum Mechanics,* (Springer-Verlag, Berlin 1992, 2nd corrected printing)

[2] W. P. Schleich, *Quantum Optics in Phase Space, (Wiley-VCH , Berlin 2001).*

[3] A Royer, Phys. Rev. A, **15**, 449–450 (1977).

[4] A. J. Bracken, H.-D. Doebner and J. G. Woods, Phys. Rev. Lett. **83**, 3758 (1999).

[5] A. J. Bracken, D. Ellinas and J. G. Woods, Acta Phys. Hung. B, Quant. Electronics, **20**, 121 (2004); J. Phys. A Math. and Gen. **36**, L297 (2003).

[6] K. Kraus *States, Effects and Operations*, (Springer, Berlin 1983).

[7] By means of the cyclic property of trace, the expectation value of an observable X, viz. $\langle X, \varepsilon(\rho)\rangle = Tr(X\varepsilon(\rho))$, on a state ρ, mapped by a CP map ε, as $\varepsilon(\rho) = \sum_i A_i \rho A_i^\dagger$, serves to define the dual map $\varepsilon^*(X) = \sum_i A_i^\dagger X A_i$, that satisfy the equation $\langle X, \varepsilon(\rho)\rangle = Tr(\rho\varepsilon^*(X))$.

[8] D. Ellinas and I. Tsohantjis, Rep. Math. Phys. **57**, 69 (2006).

[9] H Moya-Cessa and P L Knight , Phys. Rev. A **48**, 2479 (1993).

[10] W. M. de Muynck, *Foundations of Quantum Mechanics, an Empirisist Approach,* (Kluwer, Dordrecht, 2002).